1 Adeles

The most eye catching symbol in the above theorem is of course the $\mathbb{A}$. I believe this to be the Adele ring of some field so we begin by attempting to understand the construction of the Adele ring of a field. Pleasantly this merges two beautiful areas of mathematics, topology and algebra.

The journey to Adeles will be led by [CF76] with some modernising inspired by [Agh], and will take us through valuations, places and global fields.

1.1 Metric Spaces

The first abstraction we will need is that of an absolute value (or a multiplicative valuation) on a field. This is a straightforward imitation of the absolute value on $\mathbb{R}$ or $\mathbb{C}$ or a norm on a vector space

Definition. *An absolute value or (multiplicative) valuation on a field k is a function*

$$|-|: k \to \mathbb{R}_{\geq 0}$$

satisfying the following properties $\forall \alpha, \beta \in k$

- $|\alpha| = 0 \iff \alpha = 0$
- $|\alpha\beta| = |\alpha||\beta|$
- $|\alpha + \beta| \leq |\alpha| + |\beta|$

If in addition the valuation satisfies

$$\forall \alpha, \beta \in k \quad |\alpha + \beta| \leq max\{|\alpha|, |\beta|\}$$

then we call it *non-archimedean*, and otherwise we call it *archimedean*. By a theorem of Gelfand and Tornheim [CF76] all archimedean valuations are essentially the absolute value on $\mathbb{C}$, so they will not be of great interest.

Just as in basic analysis we can see that an absolute value induces a metric on k via

$$d(x, y) = |x - y|$$

and hence induces a metric topology on k. So we can think of a field with a valuation naturally as a metric topological space that is moreover a topological field [CF76]. will call this pair $(k, |-|)$, where k has the metric topology, a valuation field. Two valuations on the same field are said to be equivilent if they induce the same metric topology.

Definition. *A place is an equivalence class of valuations under this equivalence relation.*

The completion of k with respect to this metric, as a field with a valuation itself, which we will denote $\bar{k}$, is important in the description of Adele rings. There are three equivilent descriptions of it that we can give

- Explicit
- As an embedding
- Via a universal property

1.1.1 Explicit Completion

In concrete terms we could define $\bar{k}$ as the set

$$\bar{k} = \{[(x_i)_{i\in\mathbb{N}}]_\sim : x_i \in k \text{ such that } (x_i)_{i\in\mathbb{N}} \text{ is Cauchy}\}$$

where $[\ ]_\sim$ denotes the equivilence class of a Cauchy sequence under the relation

$$(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \text{ if and only if } \lim_{i\to\infty} |x_i - y_i| = 0$$

Indeed this is the metric completion of k. We then give this the valuation

$$|(x_i)|_{\bar{k}} = \lim_{i\to\infty} |x_i|_k$$

One could then show that up to isometric field isomorphism this is the unique complete field with valuation that contains k such that $\bar{k}$ is the closure of the image of k under the valuation $|-|_{\bar{k}}$.

1.1.2 Embedding Completion

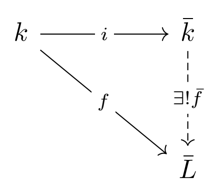
Alternitively one could define the completion of k by first proving that every valuation field can be isometrically embedded in some complete valuation field $\bar{k}$ such that $\bar{k}$ is the closure of the image of k in $\bar{k}$ under this isometric embedding. Then show that this $\bar{k}$ is unique up to field and metric isormophism.

1.1.3 Completion as a Functor

As usual when dealing with objects with a lot of information attached the language of category theory can, at least superficially, simplify the process.

We could start the process again by noting that valuation fields form a category. The morphisms are those that are most natural between a metric space which is also a topological field, isometric ring homomorphisms. Note that isometries are all continuous so we are also accounting for the topological structure too.

Then the completion of an object in this category is the universal object and arrow pair $(\bar{k}, i)$ such that $\bar{k}$ is complete, $i : k \to \bar{k}$ and any other morphism $f : k \to \bar{L}$, where $\bar{L}$ is complete, factors through $\bar{k}$ i.e.



note that a morphism in this category is always injective (because the kernel is an ideal and $1 \neq 0$ in a field) so it is justified to think of this as an embedding. Moroever we know such an object exists because we have explicitly constructed one.

We can also define another category, that of complete valuation fields. If we assign to $f : k \to \ell$ the unique lift of $i_\ell \circ f : k \to \bar{\ell}$ then we get a functor between these two categories that we may call the completion.

1.2 Local VS Global Fields

A global field is a finite extension of $\mathbb{Q}$ or the field of rational functions with Coefficients in $\mathbb{F}_q$, denoted $\mathbb{F}_q(X)$.

A local field is a valuation field that is complete wrt the valuations topology which has a finite residue field.

None of the global fields are ever local, regardless of the valuations.

why this is local vs global. 1. the global sections of a Riemann surface? 2. The ring of integers of the local one is a local ring? 3. Local rings are the completions of global fields? The last comment is true but I suspect that I have no idea why and that a good portion of understanding for the class field theory stuff might hide in there.

1.3 Adeles

The construction of the Adele ring as a topological space requires a construction called the restricted product.

Definition. *Given a family of topological spaces $\mathcal{T}_i, i \in I$ and a family of open sets $\mathcal{U}_j \subseteq \mathcal{T}_j, j \in J \subseteq I$ we define as a set*

$$\prod_{i \in I}^{res} \mathcal{T}_i = \{(x_i)_{i \in I} : x_i \in \mathcal{T}_i \text{ and for all but finitely many } j \in J, x_j \in \mathcal{U}_j\}$$

This set with the topology generated by the basis of opens

$$\left\{ \prod_{i \in I} X_i : X_i \subseteq \mathcal{T}_i \text{ open and for all but finitely many } j \in J, X_j = \mathcal{U}_j \right\}$$

is called the restricted product of $\mathcal{T}_i$ with respect to $\mathcal{U}_j$.

We are finally ready to define the Adele ring of a global field.

Given a global field k we can consider its collection of places $\{| - |_\nu\}$. Then we let k_ν be the completion of k with respect to $| - |_\nu$. We need open sets to take the restricted product over so for each non-Archimedian place define its ring of integers

$$\mathcal{O}_\nu = \{a \in k_\nu : |a| \leq 1\}$$

To see that these are open we need to see that each multiplicative valuation has a discrete image and therefore $|a| \leq 1 \iff |a| < 1 + \epsilon$ for some epsilon small, and so this is a ball and hence open in the metric topology. Then we can take the restricted product of all the k_ν with respect to these open sets

$$V_k = \prod_{\nu}^{res} k_\nu$$

which we call the Adele ring of k.

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